

Fluctuations of Energy Density and Validity of Semiclassical Gravity

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From calculations of the variance, or the fluctuations, and the mean energy density of a massless scalar field in the Minkowski vacuum as a function of an intrinsic scale defined by the world function between two nearby points (as used in point separation regularization), we show that, contrary to prior claims, the ratio of variance to its mean squared being of the order unity does not imply a failure of semiclassical gravity. It is more a consequence of the quantum nature of the state of the matter field than any inadequacy of the theory of spacetime with quantum matter as source.

1. INTRODUCTION

Recent years have seen the beginning of serious studies of the fluctuations of the energy momentum tensor (EMT) $T_{\mu\nu}$ of quantum fields in spacetimes with boundaries [1] (such as the Casimir effect [2–4]), nontrivial topology (such as imaginary-time thermal field theory), or nonzero curvature (such as the Einstein universe) [5]. A natural measure of the strength of fluctuations is χ [6], the ratio of the variance $\Delta\rho^2$ of fluctuations in the energy density (expectation value of the $\hat{\rho}^2$ operator minus the square of the mean $\hat{\rho}$ taken with respect to some quantum state) to its mean squared (square of the expectation value of $\hat{\rho}$):

$$\chi \equiv \frac{\langle \hat{\rho}^2 \rangle - \langle \hat{\rho} \rangle^2}{\langle \hat{\rho} \rangle^2} \equiv \frac{\Delta\rho^2}{\bar{\rho}^2} \quad (1.1)$$

Alternatively, we can use the quantity introduced by Kuo and Ford [4],

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$$\Delta \equiv \frac{\langle \hat{\rho}^2 \rangle - \langle \hat{\rho} \rangle^2}{\langle \hat{\rho}^2 \rangle} = \frac{\chi}{\chi + 1} \quad (1.2)$$

Assuming a positive-definite variance, $\Delta \rho^2 \geq 0$, then $0 \leq \chi \leq \infty$ and $0 \leq \Delta \leq 1$ always, with $\Delta \ll 1$ falling in the classical domain. Kuo and Ford (KF) displayed a number of quantum states (vacuum plus two-particle state, squeezed vacuum, and Casimir vacuum) with respect to which the expectation value of the energy momentum tensor (00 component) gives rise to negative local energy density. For these states Δ is of the order of unity. From this result they drew the implications, among other interesting inferences, that semiclassical gravity (SCG) [7] based on the semiclassical Einstein equation

$$G_{\mu\nu} = 8\pi G \langle \hat{T}_{\mu\nu} \rangle \quad (1.3)$$

(where $G_{\mu\nu}$ is the Einstein tensor and G the Newton gravitational constant) could become invalid under these conditions. Incorporating fluctuations of quantum fields as source elevates SCG to the level of stochastic semiclassical gravity (SSG) [8] based on the Einstein–Langevin equations [9], which is an active area of current investigation focusing on stochastic fields and metric fluctuations. The validity of semiclassical gravity in the face of fluctuations of quantum fields as source is an important issue which has caught the attention of many authors [10]. We hold a different viewpoint on this issue from KF, which we hope to clarify in this report. Details of our calculation and discussions can be found in ref. 6.

There are two groups of interrelated issues in quantum field theory in flat (ordinary QFT) or curved spacetimes (QFTCST), or semiclassical gravity (SCG, where the background spacetime dynamics is determined by the back-reaction of the mean value of quantum fields): one pertaining to quantum fields and the other to spacetimes. We discuss the first set relating to the fluctuations of the EMT over its mean values with respect to the vacuum state. It strikes us as no great surprise that states which are more ‘quantum.’ (e.g., squeezed states) in nature than classical (e.g., coherent states) (see any text in quantum optics [e.g., 11] may lead to large fluctuations comparable to the mean in the energy density. This can be seen even in the ratio of expectation values of moments of the displacement operators in simple quantum harmonic oscillators.³ Such a condition exists peacefully with the underlying spacetime at least at the low energy of today’s universe. We do not see sufficient ground to question the validity of SCG at energy below the Planck energy when the spacetime is depictable by a manifold structure, as approximated locally by the Minkowski space.

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To assess this situation we want to see what the variance of fluctuations to the mean-squared ratio of a quantum field is for the simplest case of Minkowski spacetime, i.e., good old quantum field theory. If $\Delta = O(1)$ holds also for Minkowski space, where SCG is known to be valid, it would provide a clear-cut contradiction to the criterion of KF. We find that $\Delta = 2/5$, indicating that quantum fluctuations are indeed quite large. We view this result as reflecting the quantum nature of the vacuum state and saying little about the compatibility of the field source with the spacetime the quantum field lives in.

In contrast, our view on this issue is that one should refer to a scale (of interaction or for probing accuracy) when measuring the validity of SCG. The conventional belief is that when reaching the Planck scale from below, QFTCSG will break down because, among other things happening, graviton production at that energy will become significant so as to render the classical background spacetime unstable, and the mean value of quantum fields taken as a source for the Einstein equation becomes inadequate. For this purpose we wish to introduce a scale in the spacetime points where quantum fields are defined to monitor how the mean value and the fluctuations of the energy momentum tensor change. Point separation [13, 14] would be an ideal method to adopt for this purpose. Another is by means of smeared fields [6].

In ref. 6 we derived expressions for the EM bitensor operator, its mean, and its fluctuations as functions of the point separation r or smearing distance σ , for a massless scalar field in both the Minkowski and the Casimir spacetimes. The interesting result we found is that while both the vacuum expectation value and the fluctuations of energy density grow as $r, \sigma \rightarrow 0$, the ratio of the variance of the fluctuations to its mean squared remains a constant χ_d (d is the spatial dimension of spacetime) which is independent of σ . The measure $\Delta_d [= \chi_d/(\chi_d + 1)]$ depends on the dimension of spacetime and is of the order unity. It varies only slightly for spacetimes with boundary or nontrivial topology. For example, $\Delta_{d=3}$ for Minkowski is $2/5$, while for Casimir it is $6/7$. Add to this our prior result [5] for the Einstein universe, $\Delta_{d=3} = 111/112$, independent of curvature, and that for hot flat space [12] $\Delta_{d=3} = 2/5$, and we see a general pattern emerging.

These results allow us to address two separate but interrelated issues: (a) The fluctuations of the energy density as well as its mean both increase with decreasing distance (or probing scale), and (b) the ratio of the variance of the fluctuations in EMT to its mean squared is of the order unity. We view the first, but not the second, feature as linked to the question of the validity of SCG—the case for Minkowski spacetime alone is sufficient to testify to the fallacy of Kuo and Ford's criterion. The second feature represents something quite different, pertaining more to the quantum nature of the vacuum state than to the validity of SCG.

We summarize the results of our recent calculations in Section 2 for Minkowski space and use them to discuss the above issues in Section 3. We also include the results of calculations by means of smear-field operators for a Casimir topology in the Appendix for comparison.

2. POINT-SEPARATED ENERGY DENSITY AND FLUCTUATIONS OPERATORS

For a classical (c-number) massless scalar field, the energy density is

$$\rho(t_1, \mathbf{x}_1) = \frac{1}{2}((\partial_{t_1} \phi)^2 + (\vec{\nabla} \phi)^2) \quad (2.1)$$

For quantum fields the field quantities become operators. Point separation consists of symmetrically splitting the operator product and similarly for the derivatives of the field operators,

$$\begin{aligned} (\partial_{t_1} \hat{\phi}(t_1, \mathbf{x}_1))^2 &\rightarrow \frac{1}{2}((\partial_{t_1} \hat{\phi}(t_1, \mathbf{x}_1))(\partial_{t_2} \hat{\phi}(t_2, \mathbf{x}_2)) \\ &\quad + (\partial_{t_2} \hat{\phi}(t_2, \mathbf{x}_2))(\partial_{t_1} \hat{\phi}(t_1, \mathbf{x}_1))) \end{aligned} \quad (2.2)$$

Perform a mode expansion for the field operator

$$\hat{\phi}(t_1, \mathbf{x}_1) = \int d\mu(\mathbf{k}_1) (\hat{a}_{\mathbf{k}_1} u_{\mathbf{k}_1}(t_1, \mathbf{x}_1) + \hat{a}_{\mathbf{k}_1}^\dagger u_{\mathbf{k}_1}^*(t_1, \mathbf{x}_1)) \quad (2.3)$$

with

$$u_{\mathbf{k}_1}(t_1, \mathbf{x}_1) = N_{\mathbf{k}_1} e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 - t_1 \omega_1)}, \quad \omega_1 = |\mathbf{k}_1| \quad (2.4)$$

where $\int d\mu(\mathbf{k}_1)$ is the integration measure and $N_{\mathbf{k}_1}$ are the normalization constants.

Expanding the field operators and their derivatives in normal modes in the expression for the energy density, and taking its vacuum expectation value, we obtain

$$\begin{aligned} \rho(t_1, \mathbf{x}_1; t_2, \mathbf{x}_2) &= \langle 0 | \hat{\rho}(t_1, \mathbf{x}_1; t_2, \mathbf{x}_2) | 0 \rangle \\ &= \int d\mu(\mathbf{k}_1) N_{\mathbf{k}_1}^2 \omega_1^2 \cos(\mathbf{k}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2) - (t_1 - t_2) \omega_1) \end{aligned} \quad (2.5)$$

Now consider the point-separated energy density correlation operator, $\hat{\rho}(t_1, \mathbf{x}_1, t'_1, \mathbf{x}'_1) \hat{\rho}(t_2, \mathbf{x}_2, t'_2, \mathbf{x}'_2)$, defined at pairwise points $(x_1, x'_1), (x_2, x'_2)$. A regularized energy density is obtained by taking the coincidence limit of the pairwise points. The vacuum correlation function (second cumulant) for the energy density operator is defined as

$$\begin{aligned} \Delta\rho^2(t_1, \mathbf{x}_1, t'_1, \mathbf{x}'_1; t_2, \mathbf{x}_2, t'_2, \mathbf{x}'_2) &= \langle 0|\hat{\rho}(t_1, \mathbf{x}_1; t'_1, \mathbf{x}'_1)\hat{\rho}(t_2, \mathbf{x}_2; t'_2, \mathbf{x}'_2)|0\rangle \\ &\quad - \rho(t_1, \mathbf{x}_1; t'_1, \mathbf{x}'_1) \rho(t_2, \mathbf{x}_2; t'_2, \mathbf{x}'_2) \end{aligned} \quad (2.6)$$

Since the divergences present in $\langle 0|\hat{\rho}(t_1, \mathbf{x}_1)\hat{\rho}(t_2, \mathbf{x}_2)|0\rangle$ for $(t_2, \mathbf{x}_2) \neq (t_1, \mathbf{x}_1)$ are canceled by those due to $\langle 0|\hat{\rho}(t_1, \mathbf{x}_1)|0\rangle$ and $\langle 0|\hat{\rho}(t_2, \mathbf{x}_2)|0\rangle$, we can assume $(t'_1, \mathbf{x}'_1) = (t_1, \mathbf{x}_1)$ and $(t'_2, \mathbf{x}'_2) = (t_2, \mathbf{x}_2)$ from the start. (This will be confirmed during the computation of the vacuum expectation value, without recourse to Wick's theorem.) With this understanding we can define the vacuum energy density correlation function as

$$\begin{aligned} \Delta\rho^2(t_1, \mathbf{x}_1; t_2, \mathbf{x}_2) &\equiv \langle 0|\hat{\rho}(t_1, \mathbf{x}_1)\hat{\rho}(t_2, \mathbf{x}_2)|0\rangle - \langle 0|\hat{\rho}(t_1, \mathbf{x}_1)|0\rangle\langle 0|\hat{\rho}(t_2, \mathbf{x}_2)|0\rangle \\ &= \Delta\rho^2(t_1, \mathbf{x}_1, t_1, \mathbf{x}_1; t_2, \mathbf{x}_2, t_2, \mathbf{x}_2) \end{aligned} \quad (2.7)$$

Considering just the square of the energy density operator for now, its expectation value is

$$\begin{aligned} \langle 0|\hat{\rho}^2|0\rangle &= \frac{1}{4} \int d\mu(\mathbf{k}_1, \mathbf{k}'_1, \mathbf{k}_2, \mathbf{k}'_2) N_{k_1} N_{k'_1} N_{k_2} N_{k'_2} (\mathbf{k}_1 \cdot \mathbf{k}'_1 + \omega_1 \omega'_1) (\mathbf{k}_2 \cdot \mathbf{k}'_2 + \omega_2 \omega'_2) \\ &\quad \times e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \mathbf{k}'_1 \cdot \mathbf{x}'_1 - \mathbf{k}_2 \cdot \mathbf{x}_2 - \mathbf{k}'_2 \cdot \mathbf{x}'_2) - i(t_1 \omega_1 + t'_1 \omega'_1 - t_2 \omega_2 - t'_2 \omega'_2)} \\ &\quad \times \left\{ (\delta_{\mathbf{k}_1, \mathbf{k}'_2} \delta_{\mathbf{k}'_1, \mathbf{k}_2} + \delta_{\mathbf{k}_1, \mathbf{k}_2} \delta_{\mathbf{k}'_1, \mathbf{k}'_2}) + \frac{1}{4} e^{-2i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \mathbf{k}'_1 \cdot \mathbf{x}'_1 + t_2 \omega_2 + t'_2 \omega'_2)} \delta_{\mathbf{k}_1, \mathbf{k}'_1} \delta_{\mathbf{k}_2, \mathbf{k}'_2} \right\} \\ &\quad \times \left((e^{2i(\mathbf{k}'_1 \cdot \mathbf{x}'_1 + t_1 \omega_1)} + e^{2i(\mathbf{k}_1 \cdot \mathbf{x}_1 + t_1 \omega_1)}) (e^{2i(\mathbf{k}'_2 \cdot \mathbf{x}'_2 + t_2 \omega_2)} + e^{2i(\mathbf{k}_2 \cdot \mathbf{x}_2 + t_2 \omega_2)}) \right) \\ &= \frac{1}{4} \int d\mu(\mathbf{k}_1, \mathbf{k}_2) N_{k_1}^2 N_{k_2}^2 \{ (\mathbf{k}_1 \cdot \mathbf{k}_2 + \omega_1 \omega_2)^2 \\ &\quad \times (e^{i(\mathbf{k}_1 \cdot (\mathbf{x}_1 - \mathbf{x}'_2) + \mathbf{k}_2 \cdot (\mathbf{x}'_1 - \mathbf{x}_2)) - i((t_1 - t'_2) \omega_1 + (t'_1 - t_2) \omega_2)} \\ &\quad + e^{i(\mathbf{k}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{k}_2 \cdot (\mathbf{x}'_1 - \mathbf{x}'_2)) - i((t_1 - t_2) \omega_1 + (t'_1 - t'_2) \omega_2)} \\ &\quad + [\omega_1^2 (e^{-i(\mathbf{k}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2) - (t_1 - t'_1) \omega_1)} + e^{i(\mathbf{k}_1 \cdot (\mathbf{x}_1 - \mathbf{x}'_1) - (t_1 - t'_1) \omega_1)} \\ &\quad \times \omega_2^2 (e^{-i(\mathbf{k}_2 \cdot (\mathbf{x}_2 - \mathbf{x}'_2) - (t_2 - t'_2) \omega_2)} + e^{i(\mathbf{k}_2 \cdot (\mathbf{x}_2 - \mathbf{x}'_2) - (t_2 - t'_2) \omega_2)}))] \} \end{aligned} \quad (2.8)$$

We recognize that the last two lines of the above expression is just $\rho(t_1, \mathbf{x}_1; t'_1, \mathbf{x}'_1)\rho(t_2, \mathbf{x}_2; t'_2, \mathbf{x}'_2)$. Thus, the remainder is the desired expression for $\Delta\rho^2(t_1, \mathbf{x}_1, t'_1, \mathbf{x}'_1; t_2, \mathbf{x}_2, t'_2, \mathbf{x}'_2)$. This expression is finite for $(t'_1, \mathbf{x}'_1) \rightarrow (t_1, \mathbf{x}_1)$ and $(t'_2, \mathbf{x}'_2) \rightarrow (t_2, \mathbf{x}_2)$ as long as $(t_1, \mathbf{x}_1) \neq (t_2, \mathbf{x}_2)$. Letting $(t, \mathbf{x}) = (t_2, \mathbf{x}_2) - (t_1, \mathbf{x}_1)$, our results for the energy density and its correlation function are

$$\rho(t, \mathbf{x}) = \int d\mu(\mathbf{k}_1) N_{k_1}^2 \omega_1^2 \cos(\mathbf{x} \cdot \mathbf{k}_1 - t\omega_1) \quad (2.9)$$

and

$$\begin{aligned} \Delta\rho^2(t, \mathbf{x}) &= \frac{1}{2} \int d\mu(\mathbf{k}_1, \mathbf{k}_2) N_{k_1}^2 N_{k_2}^2 (\mathbf{k}_1 \cdot \mathbf{k}_2 \\ &\quad + \omega_1 \omega_2)^2 e^{-i\mathbf{x} \cdot (\mathbf{k}_1 + \mathbf{k}_2) + i t(\omega_1 + \omega_2)} \end{aligned} \quad (2.10)$$

For a Minkowski space $R^1 \times R^d$ with d spatial dimensions the mode density is

$$\int d\mu(\mathbf{k}) = \int_0^\infty k^{d-1} dk \int_{S_d^{d-1}} d\Omega_{d-1} \quad \text{with} \quad \int_{S_d^{d-1}} d\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad (2.11)$$

and the mode function normalization constant is $N_{k_1} = 1/\sqrt{2^{d+1} \pi^d \omega_1}$. We introduce the angle between two momenta in phase space, γ , via

$$\mathbf{k}_1 \cdot \mathbf{k}_2 = k_1 k_2 \cos(\gamma) = \omega_1 \omega_2 \cos(\gamma) \quad (2.12)$$

The averages of the cosine and cosine squared of this angle over a pair of unit spheres are

$$\int_{S^{d-1}} d\Omega_1 \int_{S^{d-1}} d\Omega_2 \cos(\gamma) = 0 \quad (2.13a)$$

$$\int_{S^{d-1}} d\Omega_1 \int_{S^{d-1}} d\Omega_2 \cos^2(\gamma) = \frac{4\pi^d}{d\Gamma(d/2)^2} \quad (2.13b)$$

With these we can proceed to evaluate the point-separated energy density

$$\begin{aligned} \rho(t, \mathbf{x}) &= \int d\mu(\mathbf{k}) N_k^2 \omega^2 \cos(\mathbf{x} \cdot \mathbf{k} - t\omega) \\ &= \frac{1}{2^d \pi^{(d+1)/2} \Gamma((d-1)/2)} \\ &\quad \times \int_0^\infty \int_{-\infty}^\infty \cos(x k_x - t\sqrt{k_\perp^2 + k_x^2}) k_\perp^{d-2} \sqrt{k_\perp^2 + k_x^2} dk_x dk_\perp \end{aligned} \quad (2.14)$$

where we take $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 = x\hat{x}$ and decompose $\mathbf{k} = (k_x, \mathbf{k}_\perp)$ into one component along \hat{x} and two perpendicular to \hat{x} . We change variables to $k_x = k \cos \phi$ and $k_\perp = |\mathbf{k}_\perp| = k \sin \phi$. The final result for the point-separated energy density in Minkowski space is (restricting to odd d)

$$\rho(t, x) = -\frac{(-1)^{(d-1)/2}}{2\pi^{(d+1)/2}} \frac{(dt^2 + x^2)}{(t^2 - x^2)^{(d+3)/2}} \Gamma\left(\frac{d+1}{2}\right) \quad (2.15)$$

For the energy density correlation function, after some integrations we find

$$\Delta\rho^2(t, x) = \frac{\Gamma((d+1)/2)^2}{\pi^{d+1}} \left(\frac{4t^2x^2 + d(t^2 + x^2)^2}{(t^2 - x^2)^{d+3}} \right) \quad (2.16)$$

We can write the fluctuations in terms of the point-separated energy density as

$$\Delta\rho^2(t, x) = \chi_d(t, x)(\rho(t, x))^2 \quad (2.17)$$

and get

$$\chi_d(t, x) = \frac{d+1}{2} \left(\frac{4t^2x^2 + d(t^2 + x^2)^2}{(dt^2 + x^2)^2} \right) \quad (2.18)$$

Or,

$$\Delta_d(t, x) = \frac{(d+1)(4t^2x^2 + d(t^2 + x^2)^2)}{2(2t^2x^2 + x^4) + d^2(3t^4 + 2t^2x^2 + x^4) + d(t^4 + 10t^2x^2 + x^4)} \quad (2.19)$$

To extract physical meaning out of this for a pointwise quantum field theory, we have to work in the $(t, x) \rightarrow 0$ limit (recall $t = t_1 - t_2$, $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$, $x = |\mathbf{x}|$), for only then $\rho(t, x) \rightarrow \langle 0|\hat{\rho}|0\rangle$. Taking the limit along the time direction ($x = 0$), we get

$$\Delta_d(t, x = 0) = \frac{d+1}{1+3d} = \Delta_{d,\text{Minkowski}} \quad (2.20)$$

On the other hand, taking the limit along the spatial direction ($t = 0$), we get

$$\Delta_d(t = 0, x) = \frac{d(d+1)}{2+d+d^2} = \Delta_{d,L,\text{Reg}} \quad (2.21)$$

where $\Delta_{d,L,\text{Reg}}$ is for the regularized fluctuations of the energy density for Casimir space with periodicity L .

We can also approach this problem in another way. Since both the point-separated energy density and the correlation function have a direction dependence, we can “average” over the direction. We take the hyperspherical averging procedure. This involves first Wick rotating to imaginary time ($t \rightarrow i\tau$). Then we take the hyperspherical average in the Euclidean geometry and then Wick rotate back to Minkowski space. For the energy density

$$\rho_E(\tau, x) = \frac{\Gamma((d+1)/2)}{2\pi^{(d+1)/2}} \frac{d\tau^2 - x^2}{(\tau^2 + x^2)^{(d+3)/2}} \quad (2.22)$$

Now expressing $\tau = r \sin \theta$ and $x = r \cos \theta$, we can do the averaging

$$\begin{aligned} \rho_E(r) &= \frac{1}{2\pi} \int_0^{2\pi} \rho_E(r \sin \theta, r \cos \theta) d\theta \\ &= \frac{\Gamma((d+1)/2)}{2\pi^{(d+3)/2} r^{d+1}} \int_0^{2\pi} (d \sin^2(\theta) - \cos^2(\theta)) d\theta \\ &= \frac{(d-1) \Gamma((d+1)/2)}{4\pi^{(d-1)/2} r^{d+1}} \end{aligned} \quad (2.23)$$

Doing the same for the correlation function, we obtain

$$\begin{aligned} \Delta \rho_E^2(r) &= \frac{(d+1) \Gamma((d+1)/2)^2}{32\pi^{d+2} r^{2(d+1)}} \int_0^{2\pi} (d-1 + (d+1) \cos(4\theta)) d\theta \\ &= \frac{(d^2-1) \Gamma((d+1)/2)^2}{32\pi^{d+1} r^{2(d+1)}} \end{aligned} \quad (2.24)$$

With these results, we have

$$\chi_{d,\text{Avg}} = \frac{d+1}{2(d-1)} \quad \text{and} \quad \Delta_{d,\text{Avg}} = \frac{d+1}{-1+3d} \quad (2.25)$$

independent of whether or not we Wick rotate back to Minkowski space.

It is interesting to observe that the three sets of results depend on the direction in which the two points come together, and changes if one averages over all directions. This feature of point separation is known, but it could also reveal some properties of possible extended structure of the underlying spacetime.

3. DISCUSSIONS

Let us first display the results of our calculations for the fluctuations of the energy density and then ponder the implications of these findings for (a) fluctuations to mean ratio and the validity of semiclassical gravity, (b) the dependence of fluctuations on both the intrinsic scale (defined by smearing or point separation) and the extrinsic scale (such as the Casimir or finite-temperature periodicity), and (c) the treatment of divergences and meaning of regularization [6].

In Minkowski space we obtain different results from three different ways of taking the coincidence limit. They are given as follows:

(i) Time-direction-separated points: Hot-flat space result [12]

$$\Delta_{d,\text{Minkowski}} = \frac{d + 1}{1 + 3d} \tag{3.1}$$

with the values

$$\begin{array}{r} d \\ 3 \\ \infty \\ \frac{1}{2} \\ \frac{3}{8} \end{array} \quad \begin{array}{r} 1 \\ 5 \\ \Delta_{d,\text{Minkowski}} \\ \frac{2}{5} \\ \frac{1}{3} \end{array}$$

(ii) Space-direction-separated points: Casimir results [3–6]

$$\Delta_{d,L,\text{Reg}} \equiv \frac{\Delta \rho_{d,\text{Reg}}^2}{\Delta \rho_{L,\text{Reg}}^2 + (\rho_{L,\text{Reg}})^2} = \frac{d(d + 1)}{2 + d + d^2} \tag{3.2}$$

with the values

$$\begin{array}{r} d \\ 3 \\ \infty \\ \frac{1}{2} \\ \frac{15}{16} \end{array} \quad \begin{array}{r} 1 \\ 5 \\ \Delta_{d,L,\text{Reg}} \\ \frac{6}{7} \\ 1 \end{array}$$

(iii) Averaged Euclidean directions [6]

$$\chi_{d,\text{Avg}} = \frac{d + 1}{2(d - 1)} \quad \text{and} \quad \Delta_{d,\text{Avg}} = \frac{d + 1}{-1 + 3d} \tag{3.3}$$

with the values

$$\begin{array}{r} d \\ 3 \\ \infty \\ 1 \\ \frac{3}{7} \end{array} \quad \begin{array}{r} 1 \\ 5 \\ \Delta_{d,\text{Avg}} \\ \frac{1}{2} \\ \frac{1}{3} \end{array}$$

3.1. Fluctuation to Mean Ratio and Validity of SCG

If we adopt the criterion of Kuo and Ford [4] that the variance of the fluctuations relative to the mean squared (vev taken with respect to the

ordinary Minkowskian vacuum) being of the order unity is an indicator of the failure of SCG, then all spacetimes studied above would indiscriminately fall into that category, and SCG fails wholesale, regardless of the scale at which these physical quantities are probed. This contradicts common expectation that the SCG is valid at scales below the Planck energy. We believe that the criterion for the validity or failure of a theory in its depiction of any system in nature should depend on the range of interaction or the energy scale at which it is probed. Our findings here suggest that this is indeed the case: Both the mean (the vev of EMT with respect to the Minkowski vacuum) AND the fluctuations of EMT increase as the scale decreases. As one probes into an increasingly finer scale the expectation value of EMT grows in value and the induced metric fluctuations become important, signifying the inadequacy of SCG. A generic scale for this to happen is the Planck length. At such energy density and above, particle creation from the quantum field vacuum would become copious and their backreaction on the background spacetime would become important [7]. Fluctuations in the quantum field EMT entails these quantum processes. The induced metric fluctuations [8, 9] render the smooth manifold structure of spacetime inadequate, spacetime foams [15] including topological transitions [16] begin to appear, and SCG no longer can provide an adequate description of these dominant processes. This picture first conjured by Wheeler is consistent with the common notion adopted in SCG, and we believe it is a valid one.

3.2. Dependence of Fluctuations on Intrinsic and Extrinsic Scales

Let us now look at the bigger picture and see if we can capture the essence of the results with some general qualitative arguments. We want to see if there is any simple reason behind the following results we obtained:

- (a) $\Delta_d = O(1)$.
- (b) The specific numeric values of Δ_d for the different cases.
- (c) That Δ_d for the Minkowski space from the coincidence limit of taking a spatial point separation is identical to the Casimir case at the coincidence limit (6/7) and identical to the hot-flat space result (2/5) [12] from taking the coincidence limit of a temporal point separation.

Point (a) has also been shown by earlier calculations [4, 5], and our understanding is that this is true only for states of quantum nature, including the vacuum and certain squeezed states, but probably not true for states of a more classical nature like the coherent state. We also emphasized that this result should not be used as a criterion for the (in)validity of semiclassical gravity.

For point (b), we can trace back the calculation of the fluctuations (second cumulant) of the energy momentum tensor in ratio to its mean (first

moment) to the integral of the term containing the inner product of two momenta $\mathbf{k}_1 \cdot \mathbf{k}_2$ summed over all participating modes. The modes contributing to this are different for different geometries, e.g., Minkowski versus Casimir boundary—for the Einstein universe this enters as $3j$ symbols—and could account for the difference in the numerical values of Δ_d for the different cases.

For point (c) the difference of results between taking the coincidence limit of a spatial versus a temporal point separation is well known in QFTCS. The case of a temporal split involves integration of three spatial dimensions, while the case of a spatial split involve integration of two remaining spatial and one temporal dimension, which would give different results. The calculation of fluctuations involves the second moment of the field and is in this regard similar to what enters into the calculation of moments of inertia⁴ for rotating objects. We suspect that the difference between the temporal and the spatial results is similar, to the extent this analogy holds, to the difference in the moment of inertia of the same object but taken with respect to different axes of rotation.

It may appear surprising, as we felt when we first obtained these results, that in a Minkowski calculation the result of Casimir geometry or thermal field should appear, as both cases involve a scale—the former in the spatial dimension and the latter in the (imaginary) temporal dimension. But if we note that the results for Casimir geometry or thermal field are obtained at the coincidence (ultraviolet) limit, when the scale (infrared) of the problem does not intercede in any major way, then the main components of the calculations for these two cases would be similar to the two cases (of taking the coincident limit in the spatial and temporal directions) in Minkowski space. All of these cases are effectively devoid of scale as far as the point-defined field theory is concerned. As soon as we depart from this limit the effect of the presence of a scale shows up. The Casimir result (calculation in the Appendix) shows clearly that the boundary scale enters in a major way and the result for the fluctuations and the ratio are different from those obtained at the coincident limit. For other cases where a scale enters intrinsically in the problem such as that of a massive or non-conformally coupled field it would show a similar effect in these regards as the present cases (of Casimir and thermal field) where a periodicity condition exists (in the spatial and temporal directions, respectively). We expect a similar strong disparity between point-coincident and point-separated cases: The field theory changes its nature in a fundamental way with nontrivial physical meaning beyond this limit.

This raises another major issue brought to light in this investigation, i.e., the appearance of divergences and the meaning of regularization in the

⁴We thank A. Raval for this comment.

light of a point-separated versus a point-defined quantum field theory. Since we have the point-separated expressions of the EMT and its fluctuations we can study how they change as a function of separation or smearing. In particular we can see how divergences arise at the coincidence limit. Whether certain cross terms containing divergence have physical meaning is a question raised by the recent studies of Wu and Ford [17]. We can use these calculations to examine these issues and ask the broader question of what exactly regularization entails, where divergences arise, and how they are to be treated. The consideration of divergences in the fluctuations of EMT requires a more sophisticated rationale and reveals a deeper layer of issues pertaining to effective versus more fundamental theories. If we view ordinary quantum field theory defined at points as a low-energy limit of a theory of spacetime involving extended structures (such as string theory [e.g., 18]), then these results would shed light on their meaning and interconnections. We hope to explore this issue in future studies.

APPENDIX A. CASIMIR ENERGY AND FLUCTUATIONS

We have also calculated the energy density and its fluctuations for a Casimir space using smeared field operators assuming a Gaussian smearing function with spread σ [6]. The Casimir topology is obtained from a flat space (with d spatial dimensions, i.e., $R^1 \times R^d$) by imposing periodicity L in one of its spatial dimensions, say, z , thus endowing it with an $R^1 \times R^{d-1} \times S^1$ topology. Decomposing \mathbf{k} into a component along the periodic dimension and calling the remaining components \mathbf{k}_\perp , we get

$$\mathbf{k} = \left(\mathbf{k}_\perp, \frac{2\pi n}{L} \right) = (\mathbf{k}_\perp, ln) \quad (\text{A1a})$$

$$\omega_1 = \sqrt{k_\perp^2 + l^2 n^2} \quad (\text{A1b})$$

$$\int d\mu(\mathbf{k}) = \int_0^\infty k^{d-2} dk \int_{S^{d-2}} d\Omega_{d-2} \sum_{n=-\infty}^\infty \quad (\text{A1c})$$

$$N_{k_1} = \frac{1}{\sqrt{2^d L \pi^{d-1} \omega_1}} \quad (\text{A1d})$$

With this, we calculated the regularized energy density

$$\begin{aligned} \rho_{L,\text{reg}} &\equiv \lim_{\sigma \rightarrow 0} (\rho_L(\sigma) - \rho(\sigma)) \\ &= \frac{d\pi^{d/2} B_{d+1} \Gamma(-d/2) \Gamma(d/2)}{2(d+1) L^{d+1} \Gamma(d/2+1)} \end{aligned} \quad (\text{A2})$$

and get the usual results

$$\rho_{L,\text{reg}} \begin{matrix} d & 1 & 3 & 5 \\ -\pi/6L^2 & -\pi^2/90L^4 & -2\pi^3/945L^6 \end{matrix}$$

For the fluctuations of the energy density,

$$\begin{aligned} \Delta\rho_L^2(\sigma) &= \frac{l^2}{2^{2d+3}\pi^{2d}} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \int_0^\infty k_1^{d-2} dk_1 \int_0^\infty k_2^{d-2} dk_2 \int_{S^{d-2}} d\Omega_1 \int_{S^{d-2}} d\Omega_2 \\ &\times \frac{e^{-2\sigma^2(\omega_1^2 + \omega_2^2)}}{\omega_1\omega_2} (\cos(\gamma)k_1k_2 + l^2 n_1n_2 + \omega_1\omega_2)^2 \end{aligned} \tag{A3}$$

we obtain a regularized expression:

$$\Delta\rho_{L,\text{reg}}^2 = \chi_{d,L} (\rho_{L,\text{reg}})^2 \tag{A4}$$

where

$$\chi_{d,L} = \frac{d(d+1)}{2} \tag{A5}$$

For the dimensionless measure Δ we obtain a regularized expression:

$$\Delta_{d,L,\text{Reg}} \equiv \frac{\Delta\rho_{L,\text{Reg}}^2}{\Delta\rho_{L,\text{Reg}}^2 + (\rho_{L,\text{Reg}})^2} = \frac{d(d+1)}{2+d+d^2} \tag{A6}$$

with the values

$$\Delta_{d,L,\text{Reg}} \begin{matrix} d & 1 & 3 & 5 & \infty \\ \frac{1}{2} & \frac{6}{7} & \frac{15}{16} & 1 \end{matrix}$$

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